

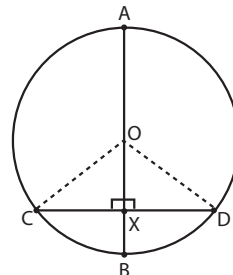
Theorem 73

1) "If a diameter of a circle is perpendicular to a chord of that circle, then that diameter bisects that chord."

3) Given: Diameter $\overline{AB} \perp$ chord \overline{CD} at X.

4) Prove: \overline{AB} bisects \overline{CD}

2)



5) Analysis: Auxiliary Lines, H.L. Congruence Postulate Corollary

6) **STATEMENT**

REASON

1. $\overline{AB} \perp \overline{CD}$ at X

1. Given

2. Draw radii \overline{OC} and \overline{OD}

2. Postulate 2

3. $\angle OXC$ and $\angle OXD$ are right angles.

3. Definition of Perpendicular Lines

4. $\triangle OXC$ and $\triangle OXD$ are right triangles

4. Definition of Right Triangle

5. $\overline{OC} \cong \overline{OD}$

5. Definition of Radii

6. $\overline{OX} \cong \overline{OX}$

6. Reflexivity of Congruence

7. $\triangle OXC \cong \triangle OXD$

7. Hypotenuse-leg Congruence Postulate Corollary

8. $\overline{CX} \cong \overline{DX}$

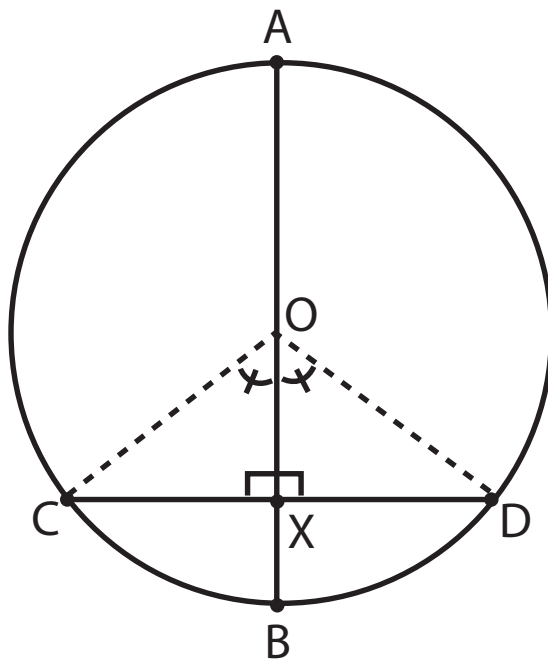
8. Definition of Similar Triangles

9. \overline{AB} bisects \overline{CD} at X

9. Definition of Segment Bisector (Q.E.D.)

COROLLARY 73a (Supporting Argument)

“If a diameter of a circle is perpendicular to a chord of that circle, then that diameter bisects the arcs intercepted by that chord.”



$$\overline{AB} \perp \overline{CD} \text{ (given)}$$

Draw Auxiliary radii \overline{OC} and \overline{OD}

$$\triangle OXC \cong \triangle OXD \text{ (H.L.)}$$

$$\angle COX \cong \angle DOX \text{ (C.P.C.T.C.)}$$

$$\widehat{CB} \cong \widehat{DB} \text{ (Theorem 65)}$$

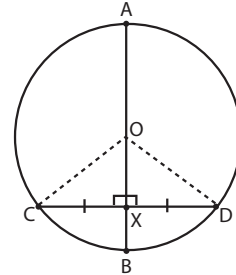
\overline{AB} bisects \widehat{CBD} (Definition of a bisector of an arc)

Theorem 74

1) "If a diameter of a circle bisects a chord, which is not a diameter of the circle, then that diameter is perpendicular to that chord."

3) Given: $\overline{CX} \cong \overline{DX}$

2)



4) Prove: $\overline{AB} \perp \overline{CD}$

5) Analysis: Auxiliary Lines, S.S.S. Congruence Postulate, Definition of Perpendicular Lines

6) **STATEMENT**

REASON

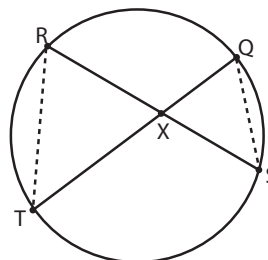
1. $\overline{CX} \cong \overline{DX}$	1. Given
2. Draw radii \overline{OC} and \overline{OD}	2. Postulate 2
3. $\overline{OX} \cong \overline{OX}$	3. Reflexivity of Congruence
4. $\overline{OC} \cong \overline{OD}$	4. Definition of Radii
5. $\triangle OXC \cong \triangle OXD$	5. S.S.S. Congruence Postulate
6. $\angle CXD$ is a straight angle	6. Definition of Straight Angle
7. $m\angle CXD$ is 180°	7. Definition of Straight Angle
8. $m\angle OXC + m\angle OXD = m\angle CXD$	8. Postulate 7 - Protractor (Angle-Addition Assumption)
9. $m\angle OXC + m\angle OXD = 180^\circ$	9. Substitution
10. $\angle OXC \cong \angle OXD$	10. Definition of Congruent Triangles
11. $m\angle OXC \cong m\angle OXD$	11. Definition of Congruent Angles
12. $m\angle OXC + m\angle OXC = 180^\circ$	12. Substitution
13. $2m\angle OXC = 180^\circ$	13. Substitution
14. $m\angle OXC = 90^\circ$	14. Multiplication of Equality
15. $\angle OXC$ is a right angle	15. Definition of Right Angle
16. $\overline{AB} \perp \overline{CD}$	16. Definition of Perpendicular Lines

Theorem 76

1) "If two chords intersect within a circle, then the product of the lengths of the segments of one chord, is equal to the product of the lengths of the segments of the other chord."

3) Given: A circle with chords \overline{RS} and \overline{TQ} intersecting at X.

2)



4) Prove: $RX \cdot SX = TX \cdot QX$

5) Analysis: Auxiliary Lines, A.A. Similarity Postulate Corollary, Definition of Similar Triangles

6) **STATEMENT**

REASON

1. Chords \overline{RS} and \overline{TQ} intersect at X.

1. Given

2. Draw chords \overline{RT} and \overline{QS}

2. Postulate 2

3. $\angle RXT \cong \angle QXS$

3. Theorem 15 - "If two lines intersect, then the vertical angles formed are congruent."

4. $\angle TRS \cong \angle SQT$

4. Corollary 67c - "If, in the same circle, two inscribed angles intercept the same, or congruent arcs, then those angles are congruent."

5. $\triangle RXT \sim \triangle QXS$

5. A.A. Similarity Postulate Corollary

6. $\frac{RX}{QX} = \frac{TX}{SX}$

6. Definition of Similar Triangles

7. $RX \cdot SX = TX \cdot QX$

7. Means-Extremes Property of Proportions

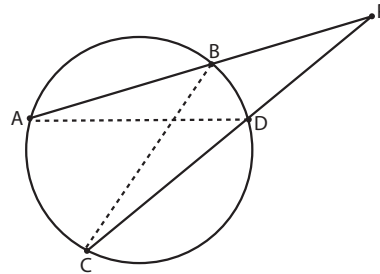
Theorem 77

1) "If two secant segments are drawn to a circle, from a single point outside the circle, then the product of the lengths of one secant segment and its external segment, is equal to the product of the lengths of the other secant segment and its external segment."

3) Given: A circle, with secant segments \overline{PA} and \overline{PC} drawn from point P.

4) Prove: $PA \cdot PB = PC \cdot PD$

2)



5) Analysis: Auxiliary Lines, Corollary 67c, Similar Triangles

6) **STATEMENT**

REASON

1. A circle with secant segments \overline{PA} and \overline{PC} drawn from point P.

1. Given

2. Draw \overline{AD} and \overline{CB}

2. Postulate 2

3. $\angle P \cong \angle P$

3. Reflexivity of Congruent Angles

4. $\angle A \cong \angle C$

4. Corollary 67c - "If in the same circle, two inscribed angles intercept the same, or congruent arcs, then those angles are congruent."

5. $\triangle PAD \sim \triangle PCB$

5. A.A. Similarity Postulate Corollary

6. $\frac{PA}{PC} = \frac{PD}{PB}$

6. Definition of Similar Triangles

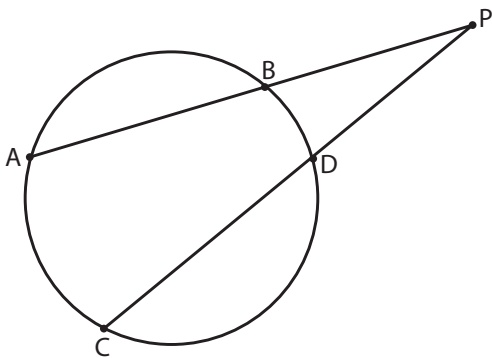
7. $PA \cdot PB = PC \cdot PD$

7. Means-Extremes Property of Proportions (Q.E.D.)

THEOREM 78

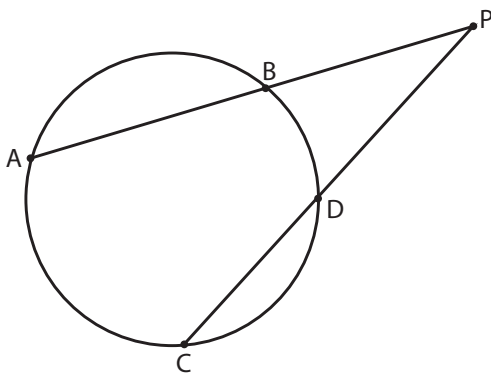
(Supporting Argument using Limits)

“If a secant segment and a tangent segment are drawn to a circle, from a single point outside the circle, then the length of that tangent segment is the mean proportional between the length of the secant segment, and the length of its external segment.”



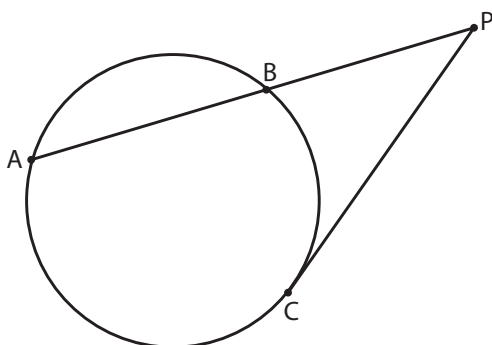
$$\frac{PA}{PB} = \frac{PC}{PD}$$

[from Theorem 77,]
 $[PA \cdot PD = PC \cdot PB]$



$$\frac{PA}{PB} = \frac{PC}{PD}$$

[from Theorem 77,]
 $[PA \cdot PD = PC \cdot PB]$



$$\frac{PA}{PB} = \frac{PC}{PD}$$

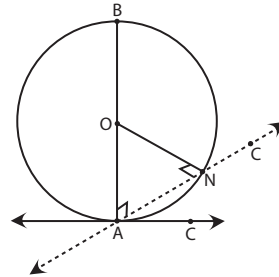
[from $PC \cong PD$]

Theorem 79

1) "If a line is perpendicular to a diameter of a circle at one of its endpoints, then the line must be tangent to the circle at that endpoint."

3) Given: $\overleftrightarrow{AC} \perp$ diameter \overline{AB} at point A.

2)



4) Prove: \overleftrightarrow{AC} must be tangent to $\odot O$ at point A.

5) Analysis: Reductio Ad Absurdum, Theorem 33, Postulate 10

6) STATEMENT

REASON

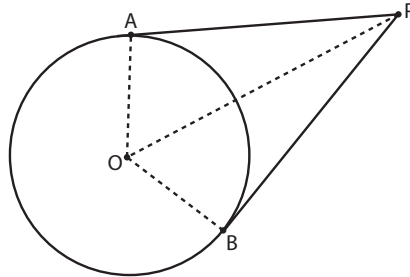
1. $\overleftrightarrow{AC} \perp$ diameter \overline{AB} at point A.	1. Given
2. Assume \overleftrightarrow{AC} is not tangent to $\odot O$ at point A.	2. Indirect Proof Assumption
3. \overleftrightarrow{AC} intersects $\odot O$ at point N.	3. Definition of Tangent Line
4. Draw radius \overline{ON}	4. Postulate 2
5. $\angle BAC$ is a right angle	5. Definition of Perpendicular Lines
6. $m\angle BAC = 90^\circ$	6. Definition of a Right Angle
7. $\angle BAC \cong \angle ONA$	7. Theorem 33 - "If two sides of a triangle are congruent, then the angles opposite them are congruent."
8. $m\angle BAC \cong m\angle ONA$	8. Definition of Congruent Angles
9. $m\angle ONA = 90^\circ$	9. Substitution
10. $\angle ONA$ is a right angle	10. Definition of Right Angle
11. $\overline{ON} \perp \overleftrightarrow{AC}$	11. Definition of Perpendicular Lines
12. But $\overline{ON} \not\perp \overleftrightarrow{AC}$	12. Postulate 10 - In a plane, through a point not on a given line, there is exactly one line perpendicular to the given line."
13. \overleftrightarrow{AC} must be tangent to $\odot O$ at point A.	13. R.A.A.

Theorem 80

1) “if two tangent segments are drawn to a circle from the same point outside the circle, then those tangent segments are congruent”

3) Given: $\odot O$ with tangent segments \overline{PA} and \overline{PB}

2)



4) Prove: $\overline{PA} \cong \overline{PB}$

5) Analysis: Auxiliary Lines, Corollary 68a,
H.L. Congruence Postulate Corollary

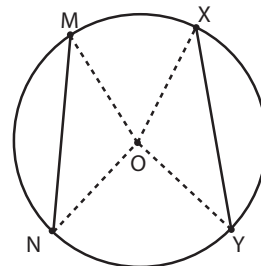
6) STATEMENT	REASON
1. $\odot O$ with tangent segments \overline{PA} and \overline{PB} .	1. Given
2. Draw \overline{OA} , \overline{OB} , and \overline{OP}	2. Postulate 2
3. $\overline{PA} \perp \overline{OA}$	3. Corollary 68a - “If, in a circle, a diameter is drawn to a tangent line, at the point of tangency, then the diameter is perpendicular to the tangent line at that point.”
4. $\overline{PB} \perp \overline{OB}$	4. Corollary 68a
5. $\angle PAO$ and $\angle PBO$ are right angles	5. Definition of Perpendicular Lines
6. $\triangle PAO$ and $\triangle PBO$ are right triangles.	6. Definition of a Right Triangle
7. $\overline{OA} \cong \overline{OB}$	7. Definition of a Radius
8. $\overline{PO} \cong \overline{PO}$	8. Reflexivity of Congruence
9. $\triangle PAO \cong \triangle PBO$	9. H.L. Congruence Postulate Corollary
10. $\overline{PA} \cong \overline{PB}$	10. Definition of Congruent Triangles (Q.E.D.)
-----	-----
11. $\angle APO \cong \angle BPO$	11. Definition of Congruent Triangles
12. PO bisects $\angle APB$	12. Definition of a Bisector of an Angle (Q.E.D.)

Theorem 81

1) "If two chords of a circle, or congruent circles, are congruent, then their intercepted minor arcs are congruent."

3) Given: In $\odot O$, $\overline{MN} \cong \overline{XY}$

2)



4) Prove: $\widehat{MN} \cong \widehat{XY}$

5) Analysis: Auxiliary Lines, Congruent Triangles, Theorem 65

6) **STATEMENT**

REASON

1. In $\odot O$, $\overline{MN} \cong \overline{XY}$

1. Given

2. Draw \overline{OM} , \overline{ON} , \overline{OX} , and \overline{OY}

2. Postulate 2

3. $\overline{OM} \cong \overline{OX}$

3. Definition of a Radius

4. $\overline{ON} \cong \overline{OY}$

4. Definition of a Radius

5. $\triangle MON \cong \triangle XOY$

5. Postulate 13 - S.S.S. Congruence

6. $\angle MON \cong \angle XOY$

6. Definition of Congruent Triangles

7. $\widehat{MN} \cong \widehat{XY}$

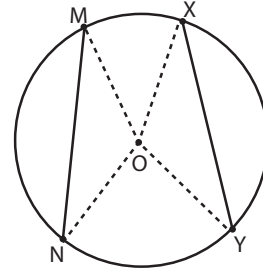
7. Theorem 65 - "If, in the same circle, or in congruent circles, two central angles are congruent, then their intercepted minor arcs are congruent." (Q.E.D.)

Theorem 82

1) "If two minor arcs of a circle, or congruent circles, are congruent, then the two chords which intercept them are congruent"

3) Given: In $\odot O$, $\widehat{MN} \cong \widehat{XY}$

2)



4) Prove: $\overline{MN} \cong \overline{XY}$

5) Analysis: Auxiliary Lines, Congruent Triangles, Theorem 66

6) **STATEMENT**

REASON

1. In $\odot O$, $\widehat{MN} \cong \widehat{XY}$

1. Given

2. Draw \overline{OM} , \overline{ON} , \overline{OX} , and \overline{OY}

2. Postulate 2

3. $\overline{OM} \cong \overline{OX}$

3. Definition of Radius

4. $\overline{ON} \cong \overline{OY}$

4. Definition of Radius

5. $\angle MON \cong \angle XOY$

5. Theorem 66 - "If two arcs of a circle are congruent, then the central angles which intercept them are congruent."

6. $\triangle MON \cong \triangle XOY$

6. S.A.S. Congruence

7. $\overline{MN} \cong \overline{XY}$

7. Definition of Congruent Triangles (Q.E.D.)

CONCURRENCY (together-run)

“I concur”

(to come together, or agree)

“current” of a river

(the running of a river)

“concurrent” events

(events which run at the same time)

“when two or more lines contain the same point”

[Note: In Geometry, because it is more significant when three or more lines contain the same point, the formal geometric definition of concurrency relates to three or more lines.]

CONCYCLIC POINTS (together-circle)

“unicycle”

(one wheel)

“cyclone”

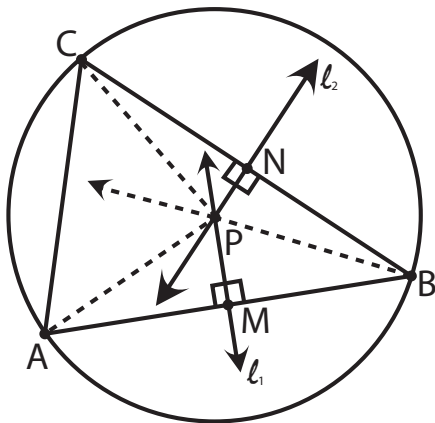
(a circular storm)

“when two or more points lie on the same circle”

[Note: In Geometry, because it is more significant when three or more points lie on the same circle, the formal geometric definition of concyclic points relates to three or more points.]

THEOREM 83 (Supporting Argument)

“If you have a triangle, then that triangle is cyclic.”



Given: $\triangle ABC$

Prove: $\triangle ABC$ is cyclic

Draw ℓ_1 perpendicular to \overline{AB} , intersecting \overline{AB} at M

ℓ_1 contains a diameter of a circle (Theorem 75)

Draw ℓ_2 perpendicular to \overline{BC} , intersecting \overline{BC} at N

ℓ_2 contains a diameter of a circle (Theorem 75)

Draw \overline{PA} , \overline{PB} and \overline{PC} (Postulate 2)

$\triangle PAM \cong \triangle PBM$ (L.L.)

$\overline{PA} \cong \overline{PB}$ (Definition of Congruent Triangles)

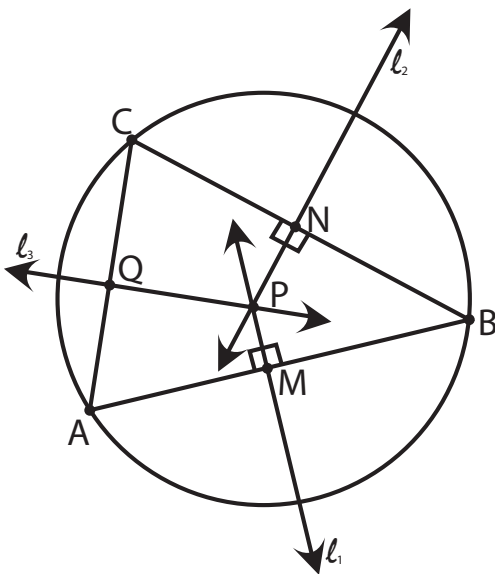
$\triangle PBN \cong \triangle PCN$ (L.L.)

$\overline{PB} \cong \overline{PC}$ (Definition of Congruent Triangles)

Draw $\odot P$ with radii \overline{PA} , \overline{PB} and \overline{PC}

COROLLARY 83a (Supporting Argument)

“If you have a triangle, then the perpendicular bisectors of the sides are concurrent.”



Given: In $\odot P$, $\overline{MP} \perp \overline{AB}$
at midpoint M.
In $\odot P$, $\overline{NP} \perp \overline{CB}$
at midpoint N.

Prove: In $\odot P$, $\overline{QP} \perp \overline{AC}$
at midpoint Q.

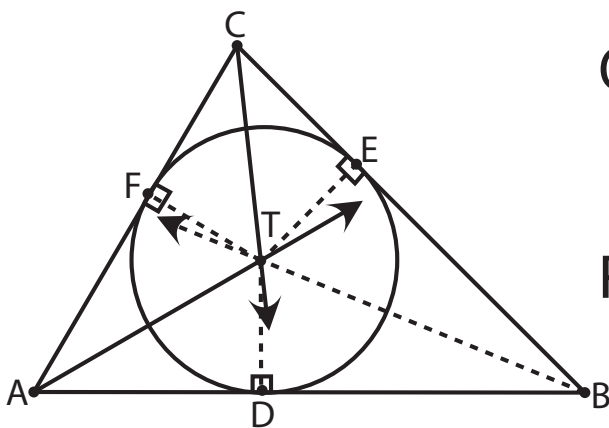
Draw l_3 through P and the midpoint Q,
of \overline{AC} (Postulate 2)

$\overline{QP} \perp \overline{AC}$ (Theorem 74)

l_1 , l_2 and l_3 are perpendicular bisectors of the
sides of $\triangle ABC$ and contain P .

COROLLARY 83b (Supporting Argument)

“If you have a triangle, then the bisectors of the angles are concurrent.”



Given: \overrightarrow{AT} bisects $\angle A$.
 \overrightarrow{CT} bisects $\angle C$

Prove: \overrightarrow{BT} bisects $\angle B$

Draw $\overline{TF} \perp \overline{AC}$ (through a point not on a given line, there exists only one line perpendicular to the given line)

Draw $\overline{TD} \perp \overline{AB}$ (through a point not on a given line, there exists only one line perpendicular to the given line)

$\triangle ATF \cong \triangle ATD$ (H.A. - Congruence Postulate Corollary 13b)

$\overline{TD} \cong \overline{TF}$ (Definition of Congruent Triangles)

$\triangle CTF \cong \triangle CTE$ (H.A. - Congruence Postulate Corollary 13b)

$\overline{TF} \cong \overline{TE}$ (Definition of Congruent Triangles)

Draw \overrightarrow{BT} (Postulate 2)

$\triangle TBD \cong \triangle TBE$ (H.L. - Congruence Postulate Corollary 13e)

$\angle \overrightarrow{TBD} \cong \angle \overrightarrow{TBE}$ (Definition of Congruent Triangles)

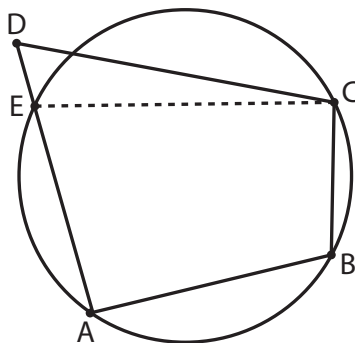
\overrightarrow{BT} bisects $\angle B$ (Definition of Bisector of an Angle)

THEOREM 84

(Supporting Argument)

“If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.”

Case 1 - Suppose one of the four points lies outside the circle.



Given: $\angle A$ is supplementary to $\angle C$
 $\angle B$ is supplementary to $\angle D$

Draw \overline{CE} (Postulate 2)

$\angle AEC$ is supplementary to $\angle B$ (Cor. 67b)

$\angle D \cong \angle AEC$ (Th. 14)

Not Possible!

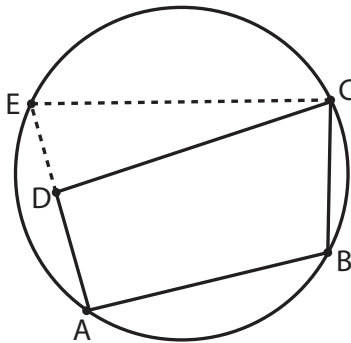
$m\angle AEC > m\angle D$ (Th. 37)

So, point D cannot be outside the circle (R.A.A.)

THEOREM 84 (cont.) (Supporting Argument)

“If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.”

Case 2 - Suppose one of the four points lies inside the circle.



Given: $\angle A$ is supplementary to $\angle C$
 $\angle B$ is supplementary to $\angle D$

Draw \overline{AE} to D (Postulate 2)

Draw \overline{EC} (Postulate 2)

$\angle E$ is supplementary to $\angle B$ (Cor. 67b)

$\angle ADC \cong \angle E$ (Th. 14)

Not Possible!

$m\angle ADC$ is $>$ $m\angle B$ (Th. 37)

So, point D cannot be inside the circle (R.A.A.)

**Therefore, point D must be on the circle,
and the quadrilateral is cyclic**